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Total Positivity of the Discrete Spline Collocation Matrix

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For an integer $k \geqslant 1$, let $\mathbf{t} := (t_i)_{-\infty}^{\infty}$ be a non-decreasing real sequence with $t_i < t_{i+k}$, and let $N_{i,k,\mathbf{t}}(x) := ([t_{i+1},..., t_{i+k}] - [t_i,..., t_{i+k-1}])(\cdot - x)_+^{k-1}$. It is well-known that $N_{i,k,\mathbf{t}}$ are B-splines of order k for the knot sequence $\mathbf{t}$. Suppose that $\boldsymbol{\mu} := (\mu_j)_{-\infty}^{\infty}$ is a sequence of integers and $\tau_j := t_{\mu_j}$. Then $N_{j,k,\boldsymbol{\tau}}$ allows the representation $N_{j,k,\boldsymbol{\tau}} = \sum_i \beta_{j,k,\boldsymbol{\tau},\mathbf{t}}(i) N_{i,k,\mathbf{t}}$. The coefficient sequence $\beta_{j,k,\boldsymbol{\tau},\mathbf{t}}$ is called a discrete B-spline with $\boldsymbol{\tau}$ and with respect to $\mathbf{t}$. This paper develops several properties of discrete B-splines and proves, in particular, the total positivity of the discrete spline collocation matrix.

Discrete polynomial splines on a uniform mesh were first introduced by Mangasarian and Schumaker [9], who defined them as solutions of certain discrete minimization problems. Later on, Schumaker [11] described constructive properties of these discrete polynomial splines. Lyche, in his thesis [8], translated many theorems on continuous polynomial splines into discrete analogues. In contrast, de Boor [2] viewed discrete B-splines as B-spline coefficients of continuous splines, which allows consideration of discrete splines for arbitrary meshes. In my opinion, de Boor's point of view has some advantages (see the postscripts). Thus, we shall develop de Boor's idea in this paper and, in particular, prove the total positivity of the discrete B-spline collocation matrix.

Let us begin with some notations. As usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ the set of real numbers, and A^B the set of functions on B into A. Thus, $\mathbb{R}^{\mathbb{Z}}$ is the set of real bi-infinite sequences. For $i, j \in \mathbb{Z}$, we mean by $[i, j]$ the set $\{n \in \mathbb{Z}; i \leqslant n \leqslant j\}$.

For $k \in \mathbb{Z}$, $k \geqslant 1$, let $\mathbf{t} := (t_i)_{-\infty}^{\infty}$ be a non-decreasing real sequence with $t_i < t_{i+k}$. It is well-known that

$$N_{i,k,\mathbf{t}} := ([t_{i+1},..., t_{i+k}] - [t_i,..., t_{i+k-1}])(\cdot - t)_+^{k-1}$$

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are B-splines of order k for the knot sequence $\mathbf{t}$. Here,

$$[\rho_0, \ldots, \rho_r] f$$

denotes the rth divided difference of the function f at the points $\rho_0, \ldots, \rho_r$, and $(x-t)_+^{k-1} := (\max\{0, x-t\})^{k-1}$.

Suppose now that $(\mu_j)_{-\infty}^{\infty}$ is an increasing sequence of integers. For $\tau_j := t_{\mu_j}$, consider the B-splines associated with the knot sequence $\boldsymbol{\tau} := (\tau_j)_{-\infty}^{\infty}$:

$$N_{j,k,\boldsymbol{\tau}} = ([\tau_{j+1}, \ldots, \tau_{j+k}] - [\tau_j, \ldots, \tau_{j+k-1}])(\cdot - t)_+^{k-1}.$$

Since $N_{j,k,\boldsymbol{\tau}}$ is also a spline with knots $\mathbf{t}$, it can be represented as a linear combination of the $N_{i,k,\mathbf{t}}$'s, by the Curry–Schoenberg theorem (see [4: p. 113]):

$$N_{j,k,\boldsymbol{\tau}} = \sum_i \beta_{j,k,\boldsymbol{\tau},\mathbf{t}}(i)\, N_{i,k,\mathbf{t}}. \tag{1}$$

Following de Boor [2], we make the following definition:

DEFINITION 1. The coefficient sequence $\beta_{j,k,\boldsymbol{\tau},\mathbf{t}} \in \mathbb{R}^{\mathbb{Z}}$ in (1) is called a discrete B-spline with knots $\boldsymbol{\tau}$ and with respect to $\mathbf{t}$.

It is known from [2] that

$$\beta_{j,k,\boldsymbol{\tau},\mathbf{t}}(i) = (t_{j+k} - t_j)[t_j, \ldots, t_{j+k}](\cdot - t_{i+1})_+ \cdots (\cdot - t_{i+k-1})_+. \tag{2}$$

When $k = 1$, (2) reads

$$\beta_{j,1,\boldsymbol{\tau},\mathbf{t}}(i) = (t_{j+1} - t_j)[t_j, t_{j+1}](\cdot - t_i)_+^0, \tag{2'}$$

where

$$\begin{aligned} (t_l - t_i)_+^0 &= 1 \quad \text{if} \quad l > i \\ &= 0 \quad \text{if} \quad l \leqslant i. \end{aligned}$$

If $\boldsymbol{\tau}$ and $\mathbf{t}$ are clear from the context, $\beta_{j,k,\boldsymbol{\tau},\mathbf{t}}$ will be abbreviated to $\beta_{j,k}$, or even to β_j.

Remark 1. Definition 1 uses a different normalization than do (5.10a) and (5.10b) of [2]. Clearly,

$$\beta_{j,k,\boldsymbol{\tau},\mathbf{t}}(i) = \frac{t_{j+k} - t_j}{t_{i+k} - t_i}\, \alpha_{\boldsymbol{\tau}}(i),$$

where $\alpha_{\boldsymbol{\tau}}(i)$ is in the sense of (5.10b) of [2].

Let us now establish some basic properties of discrete B-splines.

LEMMA 1 (Marsden's Identity).

$$\sum_j \beta_{j,k,\tau,t} = 1. \tag{3}$$

Proof. It follows from Marsden's Identity (see [2]) for continuous B-splines and (1) that

$$\sum_i N_{i,k,t} = 1 = \sum_j N_{j,k,\tau} = \sum_j \left(\sum_i \beta_{j,k,\tau,t}(i)\, N_{i,k,t}\right)$$
$$= \sum_i \left(\sum_j \beta_{j,k,\tau,t}(i)\right) N_{i,k,t}.$$

Since $N_{i,k,\tau}$ $(i \in \mathbb{Z})$ are linearly independent, (3) must hold.

LEMMA 2 (Composition formula). *Let* $\mathbf{t} := (t_i)_{-\infty}^{\infty}$ *be a non-decreasing real sequence with* $t_i < t_{i+k}$, $\boldsymbol{\rho}$ *a subsequence of* $\mathbf{t}$ *and* $\boldsymbol{\tau}$ *a subsequence of* ρ. *Then*

$$\beta_{j,k,\tau,t} = \sum_l \beta_{j,k,\tau,\rho}(l)\, \beta_{l,k,\rho,t}. \tag{4}$$

Proof. By (1),

$$\sum_i \beta_{j,k,\tau,t}(i)\, N_{i,k,t} = N_{j,k,\tau} = \sum_l \beta_{j,k,\tau,\rho}(l)\, N_{l,k,\rho}$$
$$= \sum_l \beta_{j,k,\tau,\rho}(l) \left(\sum_i \beta_{l,k,\rho,t}(i)\, N_{i,k,t}\right)$$
$$= \sum_i \left(\sum_l \beta_{j,k,\tau,\rho}(l)\, \beta_{l,k,\rho,t}(i)\right) N_{i,k,t}.$$

Since $N_{i,k,t}$ $(i \in \mathbb{Z})$ are linearly independent, (4) follows from the above equality.

LEMMA 3 (Recurrence relation). *For* $k \geqslant 2$,

$$\beta_{j,k}(i) = (\tau_{j+k} - t_{i+k-1}) \frac{\beta_{j+1,k-1}(i)}{\tau_{j+k} - \tau_{j+1}} + (t_{i+k-1} - \tau_j) \frac{\beta_{j,k-1}(i)}{\tau_{j+k-1} - \tau_j}. \tag{5}$$

Proof. Note that for any $t \in \mathbf{t}$,

$$\begin{aligned}(\tau - t_{i+1})_+ \cdots (\tau - t_{i+k-2})_+ (\tau - t_{i+k-1})_+ \\ = [(\tau - t_{i+1})_+ \cdots (\tau - t_{i+k-2})_+][(\tau - t_{i+k-1})].\end{aligned}$$

Applying Leibniz's formula to the above product, we obtain

$$\begin{aligned}\beta_{j,k}(i) &= (\tau_{j+k} - \tau_j)[\tau_j, \ldots, \tau_{j+k}](\cdot - t_{i+1})_+ \cdots (\cdot - t_{i+k-1})_+ \\ &= (\tau_{j+k} - \tau_j) \sum_{r=j}^{j+k} \{[\tau_j, \ldots, \tau_r](\cdot - t_{i+1})_+ \cdots (\cdot - t_{i+k-2})_+ \\ &\qquad \times [\tau_r, \ldots, \tau_{j+k}](\cdot - t_{i+k-1})\} \\ &= ((\tau_{j+k} - \tau_j)[\tau_j, \ldots, \tau_{j+k}])(\cdot - t_{i+1})_+ \cdots (\cdot - t_{i+k-2})_+ (\tau_{j+k} - t_{i+k-1}) \\ &\qquad + (\tau_{j+k} - \tau_j)[\tau_j, \ldots, \tau_{j+k-1}](\cdot - t_{i+1})_+ \cdots (\cdot - t_{i+k-2})_+ \\ &= (\tau_{j+k} - t_{i+k-1})\{[\tau_{j+1}, \ldots, \tau_{j+k}] - [\tau_j, \ldots, \tau_{j+k-1}]\} \\ &\qquad (\cdot - t_{i+1})_+ \cdots (\cdot - t_{i+k-2})_+ \\ &\qquad + (\tau_{j+k} - \tau_j)[\tau_j, \ldots, \tau_{j+k-1}](\cdot - t_{i+1})_+ \cdots (\cdot - t_{i+k-2})_+ \\ &= \{(\tau_{j+k} - t_{i+k-1})[\tau_{j+1}, \ldots, \tau_{j+k}] + (t_{i+k-1} - \tau_j)[\tau_j, \ldots, \tau_{j+k-1}]\} \\ &\qquad (\cdot - t_{i+1})_+ \cdots (\cdot - t_{i+k-2}) \\ &= (\tau_{j+k} - t_{i+k-1}) \frac{\beta_{j+1,k-1}(i)}{\tau_{j+k} - \tau_{j+1}} + (t_{i+k-1} - \tau_j) \frac{\beta_{j,k-1}(i)}{\tau_{j+k-1} - \tau_j}.\end{aligned}$$

This proves Lemma 3.

LEMMA 4. *For a fixed $s \in \mathbb{Z}$, let*

$$\begin{aligned} v_l &:= l \qquad &&\text{if} \quad l < s \\ &:= l - 1 &&\text{if} \quad l \geqslant s. \end{aligned}$$

$\rho_l := t_{v_l}$ *and* $\boldsymbol{\rho} := (\rho_l)_{-\infty}^{\infty}$. *In other words,* $\boldsymbol{\rho}$ *is formed by dropping an entry from* $\mathbf{t}$. *Then*

$$\beta_{l,k,\boldsymbol{\rho},\mathbf{t}}(i) = 0 \qquad \textit{for } l < i - 1 \textit{ or } l > i; \tag{6a}$$

$$\beta_{i-1,k,\boldsymbol{\rho},\mathbf{t}}(i) \geqslant 0 \qquad \textit{with strict inequality iff } t_{i+k} > t_s; \tag{6b}$$

$$\beta_{i,k,\boldsymbol{\rho},\mathbf{t}}(i) \geqslant 0 \qquad \textit{with strict inequality iff } t_i < t_s. \tag{6c}$$

Proof. If $t_l \geqslant t_s$, then $N_{l,k,\boldsymbol{\rho}} = N_{l+1,k,\mathbf{t}}$, and it follows that

$$\beta_{l,k,\boldsymbol{\rho},\mathbf{t}}(i) = \delta_{i,l+1} \qquad \text{for all } i, l.$$

In the same way, for $t_{l+k+1} \leqslant t_s$,

$$\beta_{l,k,\rho,t}(i) = \delta_{i,l} \qquad \text{for all } i, l.$$

Now (6a) is easily derived from what has been proved. Moreover, when $t_l < t_s < t_{l+k+1}$, we have

$$N_{l,k,\rho} = \beta_{l,k,\rho,t}(l) N_{l,k,t} + \beta_{l,k,\rho,t}(l+1) N_{l+1,k,t}.$$

It is known that $N_{l,k,\rho}$ has the same sign as $\beta_{l,k,\rho,t}(l)$ in $(t_l, t_l + \varepsilon)$ for sufficiently small $\varepsilon > 0$ (see [2]), so

$$\beta_{l,k,\rho,t}(l) > 0 \qquad \text{for} \quad t_l < t_s < t_{l+k+1};$$

similarly

$$\beta_{l,k,\rho,t}(l+1) > 0 \qquad \text{for} \quad t_l < t_s < t_{l+k+1}.$$

Summarizing these facts we get (6b) and (6c).

LEMMA 5. *Suppose* $\mu \in \mathbb{Z}^{\mathbb{Z}}$ *and* $\tau_j = t_{\mu_j}$ *for all* $j \in \mathbb{Z}$. *Then*

$$\beta_{j,k,\tau,t}(i) \geqslant 0$$

with equality if and only if one of the following four cases occurs:

$$t_i < t_{\mu_j}; \tag{7a}$$

$$t_i = t_{\mu_j} \quad \textit{and} \quad \max\{p \mid t_{i+p} = t_i\} > \max\{q \mid t_{\mu_{j+q}} = t_{\mu_j}\}; \tag{7b}$$

$$t_{i+k} > t_{\mu_{j+k}}; \tag{7c}$$

$$t_{i+k} = t_{\mu_{j+k}} \quad \textit{and} \quad \max\{p \mid t_{i+k-p} = t_{i+k}\} > \max\{q \mid t_{\mu_{j+k-q}} = t_{\mu_{j+k}}\}. \tag{7d}$$

Proof. We use the linear functional λ_i given by the rule:

$$\lambda_i f := \sum_{r=0}^{k-1} (-1)^{k-1-r} \Psi^{(k-1-r)}(\xi)\, D^r f(\xi), \qquad \text{all } f, \tag{8}$$

where $\Psi(t) := (t_{i+1} - t) \cdots (t_{i+k+1} - t)/(k-1)!$ and $t_i < \xi < t_{i+k}$. By the de Boor–Fix Theorem (see [4, pp. 116–118]),

$$\lambda_i N_{j,k,\tau} = \beta_{j,k,\tau,t}(i). \tag{9}$$

In case (7a), choose ξ so that $t_i < \xi < t_{\mu_j}$. Then

$$N_{j,k,\tau}(\xi) = N'_{j,k,\tau}(\xi) = \cdots = N^{(k-1)}_{j,k,\tau}(\xi) = 0.$$

Hence (8) and (9) yield

$$\beta_{j,k,\tau,t}(i) = \lambda_i N_{j,k,\tau} = 0.$$

In case (7b) write $c := \max\{p \mid t_{i+p} = t_i\}$, $d := \max\{q \mid t_{\mu_{j+q}} = t_{\mu_j}\}$. Then $d \leqslant c-1$ and

$$N_{j,k,\tau}(t_i+) = N'_{j,k,\tau}(t_i+) = \cdots = N^{(k-d-2)}_{j,k,\tau}(t_i+) = 0,$$
$$\Psi(t_i+) = \cdots = \Psi^{(c-1)}(t_i+) = 0.$$

Taking $\xi = t_i+$ in (8) and substituting these values into (8), we obtain

$$\beta_{j,k,\tau,t}(i) = \lambda_i N_{j,k,}\tau = 0.$$

Cases (7c) and (7d) can be treated in the same way.

Now suppose that none of (7a)–(7d) is true. We want to show $\beta_{j,k,\tau,t}(i) > 0$. Let

$$E := \{l \mid \mu_j \leqslant l \leqslant \mu_{j+k}, l \in \boldsymbol{\mu}\} \qquad \text{and} \qquad |E| := \text{the cardinality of } E.$$

We shall proceed by induction of $|E|$. The case $|E| = 0$ is trivial. The case $|E| = 1$ is reduced to Lemma 4. Assume now that our statement is true for $|E| < n$. We want to prove our statement is also true for $|E| = n$. Take any $s \in E$. Let $\boldsymbol{\rho}$ be defined as in Lemma 4; that is, $\nu_l = l$ for $l < s$, $\nu_l = l+1$ for $l \geqslant s$ and $\rho_l := t_{\nu_l}$. By Lemmas 2 and 4,

$$\beta_{j,k,\tau,t}(i) = \sum_l \beta_{j,k,\tau,\rho}(l)\, \beta_{l,k,\rho,t}(i)$$
$$= \beta_{j,k,\tau,\rho}(i-1)\, \beta_{i-1,k,\rho,t}(i) + \beta_{j,k,\tau,\rho}(i)\, \beta_{i,k,\rho,t}(i).$$

All terms that appear in the above equality are nonnegative. It seems appropriate to treat the following three possible subcases individually.

(i) $t_i \geqslant t_s$. In this case, $\beta_{i-1,k,\rho,t}(i) > 0$ by (6b). We need to show $\beta_{j,k,\tau,\rho}(i-1) > 0$. If $i-1 \geqslant s$, then $\nu_{i-1} = i$ and $\nu_{i-1+k} = i+k$; so $\beta_{j,k,\tau,\rho}(i-1) > 0$ by induction hypohesis. Assume now $i-1 < s$. If $t_i > t_{\mu_j}$, then $t_{i-1} > t_{\mu_j}$ or

$$t_{i-1} = t_{\mu_j} \qquad \text{and} \qquad \max\{p \mid t_{i+p-1} = t_{i-1}\} = 0 \leqslant \max\{q \mid t_{\mu_{j+q}} = t_{\mu_j}\}.$$

Hence $\beta_{j,k,\tau,\rho}(i-1) > 0$ by induction hypothesis again. Finally, suppose $t_i = t_{\mu_j}$. Then $t_i \geqslant t_s \geqslant t_{\mu_j}$ implies $t_s = t_i$. Thus $\mu_j < i$; for otherwise $\mu_j \geqslant i$ and $i \leqslant s$ would imply

$$\max\{p \mid t_{i+p} = t_i\} > \max\{q \mid t_{\mu_{j+q}} = t_{\mu_j}\},$$

a contradiction. In conclusion,

$$\max\{p \mid \rho_{i-1+p} = \rho_{i-1}\} = \max\{p \mid t_{i+p} = t_i\} \leqslant \max\{q \mid t_{\mu_j} = t_{\mu_{j+q}}\},$$

so that $\beta_{j,k,\boldsymbol{\tau},\boldsymbol{\rho}}(i-1) > 0$.

(ii) $t_{i+k} \leqslant t_s$. This case can be treated in the same way as (i) is.

(iii) $t_i < t_s < t_{i+k}$. Lemma 4 tells us that both $\beta_{i-1,k,\boldsymbol{\rho},\mathbf{t}}(i)$ and $\beta_{i,k,\boldsymbol{\rho},\mathbf{t}}(i)$ are positive in this case. Thus we need to show that at least one of $\beta_{j,k,\boldsymbol{\tau},\boldsymbol{\rho}}(i-1)$ and $\beta_{j,k,\boldsymbol{\tau},\boldsymbol{\rho}}(i)$ is positive. If either $t_i > t_{\mu_j}$ or $t_{i+k} < t_{\mu_{j+k}}$, then this holds by the observation made in (i). Next, suppose $t_i = t_{\mu_j}$, $t_{i+k} = t_{\mu_{j+k}}$ and either

$$\max\{p \mid t_{i+p} = t_i\} < \max\{q \mid t_{\mu_{j+q}} = t_{\mu_j}\}$$

or

$$\max\{p \mid t_{i+k-p} = t_i\} < \max\{q \mid t_{\mu_{j+k-q}} = t_{\mu_{j+k}}\};$$

then one can easily get $\beta_{j,k,\boldsymbol{\tau},\boldsymbol{\rho}}(i-1) > 0$ or $\beta_{j,k,\boldsymbol{\tau},\boldsymbol{\rho}}(i) > 0$, using the same argument as in (i). The remaining case to be discussed is

$$\max\{p \mid t_{i+p} = t_i\} = \max\{q \mid t_{\mu_{j+q}} = t_i\}$$

and

$$\max\{p \mid t_{i+k-p} = t_{i+k}\} = \max\{q \mid t_{\mu_{j+k-q}} = t_{i+k}\}.$$

Let $c := \max\{p \mid t_{i+p} = t_i\}$, $e := \max\{p \mid t_{i+k-q} = t_{i+k}\}$. Then

$$\mu_{j+c+1} > i + c, \qquad \mu_{j+k-e-1} < i + k - e;$$

hence

$$\begin{aligned}\mu_{j+k-e-1} - \mu_{j+c+1} &\leqslant (i+k-e-1) - (i+c+1)\\ &= (j+k-e-1) - (j+c+1).\end{aligned}$$

This means $s \in \boldsymbol{\mu}$, which contradicts the choice of s. Lemma 5 is proved.

We are now in a position to prove our main result.

THEOREM 1. *Let* $\mathbf{t} := (t_i)_{-\infty}^{\infty}$ *be a non-decreasing real sequence with* $t_i < t_{i+k}$, *all* k, $(\mu_j)_{-\infty}^{\infty}$ *an increasing integer sequence,* $\tau_j := t_{\mu_j}$ *and let* $\boldsymbol{\tau} := (\tau_j)_{-\infty}^{\infty}$. *Let* $(\beta_j)_{j=-\infty}^{\infty}$ *be the sequence of discrete B-splines of order* k *with the knot sequence* $\boldsymbol{\tau}$ *and with respect to* $\mathbf{t}$. *Let*

$$i_1 < i_2 < \cdots < i_m$$

be a finite increasing subsequence of integers, and set

$$U := (u_{rj}) := (\beta_j(i_r))_{1 \leqslant r \leqslant m}.$$

Then for every subsequence $q_1 < \cdots < q_m$,

$$\det U \begin{bmatrix} i_1, \ldots, i_m \\ q_1, \ldots, q_m \end{bmatrix} \geqslant 0 \tag{10}$$

with strict inequality iff both of the following conditions are satisfied:

(i) $\beta_{q_r}(i_r) > 0$ *for all* $r = 1, 2, \ldots, m$.

(ii) *If there is some* $s \in \boldsymbol{\mu}$ *such that* $t_{i_r} = t_s$ *for some* r, *then*

$$i_{r-d_r} < i_r - d_r,$$

where

$$d_r := k - \max\{p : t_{i_r+p} = t_{i_r}\}.$$

Proof. Write

$$A := U \begin{bmatrix} i_1, \ldots, i_m \\ q_1, \ldots, q_m \end{bmatrix}.$$

If $\beta_{q_r}(i_r) = 0$ for some r, then $\beta_{q_j}(i_l) = 0$ for all l, j, with $1 \leqslant l \leqslant r \leqslant j \leqslant m$, by Lemma 5. Thus columns $r, \ldots, m$ of A are linearly dependent and $\det A = 0$. Without loss of generality we may assume further that both the first super-diagonal and subdiagonal of A are positive,

$$\beta_{q_r}(i_{r+1}) > 0, \ r = 1, \ldots, m-1, \qquad \text{and} \qquad \beta_{q_r}(i_{r-1}) > 0, \ r = 2, \ldots, m. \tag{11}$$

Otherwise, we would have, say, $\beta_{q_{r+1}}(i_r) = 0$ for some r. It would follow that $\beta_{q_j}(i_l) = 0$ for any l, j with $1 \leqslant l \leqslant r < j \leqslant m$. Thus

$$\det A = \det U \begin{bmatrix} i_1, \ldots, i_r \\ q_1, \ldots, q_r \end{bmatrix} \cdot \det U \begin{bmatrix} i_{r+1}, \ldots, i_m \\ q_{r+1}, \ldots, q_m \end{bmatrix},$$

where

$$\det U \begin{bmatrix} i_1, \ldots, i_r \\ q_1, \ldots, q_r \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} i_{r+1}, \ldots, i_m \\ q_{r+1}, \ldots, q_m \end{bmatrix}$$

are lower order determinants of the same form. If $m = 1$, then A is a 1×1 matrix and $\det A > 0$, trivially. Thus if we use induction on m, then $\det A$ would already have the property declared in Theorem 1. From now on we always assume (11) to hold (cf. [3]).

We point out that (11) yields

$$\beta_{q_r}(i_r \pm 1) > 0, \qquad r = 1,\dots, m. \tag{12}$$

The only thing we have to prove is $\beta_{q_1}(i_1 - 1) > 0$ and $\beta_{q_m}(i_m + 1) > 0$, while in all the other cases this is a direct consequence of (11). Since $\beta_{q_2}(i_1) > 0$, $i_1 \geqslant \mu_{q_2}$, hence $i_1 - 1 \geqslant \mu_{q_2} - 1 \geqslant \mu_{q_1}$; then $\mu_{q_1,k}(i_1 - 1) > 0$, obviously. If $t_{i_1-1} = t_{\mu_{q_1}}$ and $t_{i_1-1} < t_{i_1}$, then

$$\max\{p \mid t_{i_1-1} = t_{i_1-1+p}\} = 0 \leqslant \max\{q \mid t_{\mu_{q_1}+q} = t_{i_1-1}\}.$$

So we also have $\beta_{q_1}(i_1 - 1) > 0$. The last possible case is $t_{i_1-1} = t_{\mu_{q_1}} = t_{i_1}$. Then

$$\begin{aligned}\max\{p \mid t_{i_1-1} = \cdots = t_{i_1-1+p}\} &= 1 + \max\{p \mid t_{i_1} = \cdots = t_{i_1+p}\}\\ &\leqslant 1 + \max\{q \mid t_{\mu_{q_2}} = \cdots = t_{\mu_{q_2}+q}\} \leqslant \max\{q \mid t_{\mu_{q_1}} = \cdots = t_{\mu_{q_1}+q}\};\end{aligned}$$

therefore $\beta_{q_1}(i_1 - 1) > 0$. Similarly, $\beta_{q_m}(i_m + 1) > 0$.

As in Lemma 5, let

$$E := \{l \mid \mu_q \leqslant l \leqslant \mu_{q_m+k}, l \in \boldsymbol{\mu}\}.$$

We will proceed by induction on $|E|$. If $|E| = 0$, then A is a diagonal matrix, so the proof is trivial. Suppose now that, for $|E| < n$, our theorem is proved, and we want to show the conclusion of Theorem 1 also holds for $|E| = n$.

We have proved that if (i) is violated, then $\det A = 0$. Suppose now that (ii) does not hold. Then there is some $s \in \boldsymbol{\tau}$ such that $t_{i_r} = t_s$ and $i_{r-d_r} = i_r - d_r$. Form $\boldsymbol{\rho}$ by dropping s from $\mathbf{t}$ as we did in Lemma 4. Let

$$V := (\beta_{l,k,\boldsymbol{\rho},\mathbf{t}}(i_r))_{\substack{1 \leqslant r \leqslant m\\ \mu_{q_1} \leqslant l \leqslant \mu_{q_m}+k-1}},$$

$$W := (\beta_{q_r,k,\boldsymbol{\tau},\boldsymbol{\rho}}(l))_{\substack{\mu_{q_1} \leqslant l \leqslant \mu_{q_m}+k-1\\ 1 \leqslant r \leqslant m}}.$$

Then $A = VW$ by Lemma 2. Further, the Cauchy–Binet formula (see [6]) gives

$$\det A = \sum_{\xi_1 < \xi_2 < \cdots < \xi_m} \det V \begin{bmatrix} i_1, i_2,\dots, i_m \\ \xi_1, \xi_2,\dots, \xi_m \end{bmatrix} \det W \begin{bmatrix} \xi_1, \xi_2,\dots, \xi_m \\ i_1, i_2,\dots, i_m \end{bmatrix}. \tag{13}$$

Since $t_{i_r} = t_s$ and $t_{i_r-d_r+k} = t_{i_r} - d_r$, we have

$$\beta_{i_r,k,\boldsymbol{\rho},\mathbf{t}}(i_r) = 0 \qquad \text{and} \qquad \beta_{i_r-d_r-1,k,\boldsymbol{\rho},\mathbf{t}}(i_{r-d_r}) = 0$$

by Lemma 4. Furthermore,

$$\beta_{j,k,\boldsymbol{\rho},\mathbf{t}}(u_h) = 0 \qquad \text{for } j \leqslant i_r - d_r - 1 \text{ or } j \geqslant i_r,\ h = i_r - d_r, \dots, i_r.$$

Consider the following matrix with $d_r + 1$ rows:

$$\begin{bmatrix} \cdots \beta_{i_r - d_r - 1,k,\boldsymbol{\rho},\mathbf{t}}(i_{r-d_r}) & \beta_{i_r - d_r,k,\boldsymbol{\rho},\mathbf{t}}(i_{r-d_r}) & \cdots & \beta_{i_r,k,\boldsymbol{\rho},\mathbf{t}}(i_{r-d_r}) \cdots \\ \vdots & \vdots & & \vdots \\ \cdots \beta_{i_r - d_r - 1,k,\boldsymbol{\rho},\mathbf{t}}(i_r) & \beta_{i_r - d_r,j,\boldsymbol{\rho},\mathbf{t}}(i_r) & \cdots & \beta_{i_r,k,\boldsymbol{\rho},\mathbf{t}}(i_r) \quad \cdots \end{bmatrix}.$$

All its entries except those in columns $i_r - d_r, \dots, i_r - 1$ are zero. Thus the rank of this matrix is no bigger than d_r. Hence the $d_r + 1$ rows of this matrix are linearly dependent. This shows that the rows $r - d_r$, $r - d_r + 1, \dots, r$ of the matrix A are linearly dependent. Thus the rows $r - d_r$, $r - d_r + 1, \dots, r$ of each

$$V\begin{bmatrix} i_1, i_2, \dots, i_m \\ \xi_1, \xi_2, \dots, \xi_m \end{bmatrix}$$

are linearly dependent, so that

$$\det V\begin{bmatrix} i_1, i_2, \dots, i_m \\ \xi_1, \xi_2, \dots, \xi_m \end{bmatrix} = 0 \qquad \text{for all } \xi_1 < \xi_2 < \cdots < \xi_m. \tag{14}$$

Therefore $\det A = 0$ by (13).

Suppose now that both conditions (i) and (ii) are satisfied. We want to show $\det A > 0$. We shall argue by induction on $|E|$ again. Take $s \in E$. Form $\mathbf{v}$ and $\boldsymbol{\rho}$ as we did in Lemma 4. Let V and W have the same meaning as above. By induction hypothesis and Lemma 4, all products that appear on the right-hand side of (13) are nonnegative. Let r be the least integer such that $t_{i_r} \geqslant t_s$. Then $t_{i_{r-1}} < t_s$. There are two possibilities to be discussed:

(α) $i_{r-1} < i_r - 1$.

In this case, we choose

$$\xi_h := i_h - 1 \qquad \text{for} \quad h \geqslant r$$
$$:= i_h \qquad \text{for} \quad h < r.$$

Then $\xi_1 < \xi_2 < \cdots < \xi_m$. By Lemma 4 and the choice of the ξ's,

$$\beta_{\xi_h,k,\boldsymbol{\rho},\mathbf{t}}(i_h) > 0, \qquad h = 1, 2, \dots, m.$$

In addition, if $h < r$, then we have $v_{\xi_h} = i_h$ and $v_{\xi_{h+k}} = i_{h+k}$ or i_{h+k+1}. Thus (12) together with Lemma 5 tells us that

$$\beta_{q_h,k,\tau,\mathbf{p}}(\xi_h) > 0.$$

Similarly, if $h \geqslant r$, then we have $v_{\xi_h} = i_h - 1$ and $v_{\xi_{h+k}} = i_{h+k} - 1$ or i_{h+k} so the above inequality also holds. By induction hypothesis we assert that

$$\det V \begin{bmatrix} i_1, i_2, \ldots, i_m \\ \xi_1, \xi_2, \ldots, \xi_m \end{bmatrix} > 0 \qquad \text{and} \qquad \det W \begin{bmatrix} \xi_1, \xi_2, \ldots, \xi_m \\ q_1, q_2, \ldots, q_m \end{bmatrix} > 0. \quad \text{(14a)}$$

By (13), (14) and (14a) we have

$$\det U \begin{bmatrix} i_1, i_2, \ldots, i_m \\ q_1, q_2, \ldots, q_m \end{bmatrix} > 0.$$

(β) $i_r - 1 = i_{r-1}$. In this case, condition (ii) gives

$$i_{r-d_r} < i_r - d_r,$$

where $d_r := k - \max\{p \mid t_{i+p} = t_{i_r}\}$. There exists an integer c, $1 \leqslant c \leqslant d_r$, such that $i_{r-(c-1)} = i_r - (c-1)$ but $i_{r-c} < i_r - c$. Thus

$$i_{r-c} < i_r - c = (i_r - c + 1) - 1 = i_{r-c+1} - 1. \quad (15)$$

Let

$$\begin{aligned} \xi_h &:= i_h - 1 \qquad &&\text{for} \quad h \geqslant r - c + 1 \\ &:= i_h &&\text{for} \quad h \leqslant r - c. \end{aligned} \quad (16)$$

From (15) and (16) we see that $\xi_1 < \xi_2 < \cdots < \xi_m$. Now Lemma 4 yields that

$$\det V \begin{bmatrix} i_1, i_2, \ldots, i_m \\ \xi_1, \xi_2, \ldots, \xi_m \end{bmatrix} > 0.$$

Using the same argument as in (α), we get

$$\det W \begin{bmatrix} \xi_1, \xi_2, \ldots, \xi_m \\ q_1, q_2, \ldots, q_m \end{bmatrix} > 0.$$

This proves our theorem.

Remark 2. If $\mathbf{t}$ is a strictly increasing sequence, Lemma 5 can be stated as follows:

$$\beta_{j,k,\tau,\mathbf{t}}(i) \geqslant 0$$

with strict inequality iff $t_i \leqslant t_j$, $t_{i+k} \leqslant t_{j+k}$. Furthermore, in Theorem 1, condition (ii) is automatically fulfilled as long as (i) holds.

Remark 3. It is interesting that the Schoenberg–Whitney Theorem (see [10]) can be derived from our Theorem 1. Indeed, let $\boldsymbol{\tau} = (\tau_j)_{-\infty}^{\infty}$ be a nondecreasing knot sequence, $\tau_1 < \tau_2 < \cdots < \tau_m$, and let $\mathbf{t}$ be a knot sequence formed by adding some knots to $\boldsymbol{\tau}$ so that $\mathbf{t}$ has exactly k multiples at each τ_i, $i = 1, 2, \ldots, m$. Then

$$N_{j,k,\boldsymbol{\tau}}(\tau_i) = \beta_{j,k,\boldsymbol{\tau},\mathbf{t}}(\tau_i)$$

according to (1). Now one could easily see that the Schoenberg–Whitney Theorem is a consequence of Theorem 1.

POSTSCRIPT

This work was done in July 1980. Later I became aware of the three related papers [1, 5, 7]. Essentially, whether explicitly or implicitly, these three papers view discrete B-splines as the coefficient sequences associated with the expansion of continuous polynomial splines in B-splines. This is just de Boor's point of view (see [2]). In [1], the author provided an algorithm for further subdivision of a knot sequence. The basic idea of [1] is to investigate what happens when one inserts new knots into a given knot sequence. The essential idea of the present paper is also "inserting new knots" and "inserting one new knot each time." In [5], the authors develop more properties of discrete splines. Lemmas 1 and 3 and parts of Lemmas 4 and 5 of this paper overlap with [5]. However, [5] is based on the recurrence formula, while my Theorem 1 does not need the recurrence formula though the proof for the recurrence formula (Lemma 3) is more straightforward in my opinion. In [7], the authors give the shortest way to prove the variation diminishing property of B-spline approximation by using a geometric observation. Their methods can be easily carried but to prove that the associated discrete spline collocation matrix is sign regular, but it seems hard to determine which minor is really positive along this way. In the present paper, by the composition formula (Lemma 2) and the Cauchy–Binet formula, we are able to obtain the exact criterion for the positivity of a given minor. I believe that the determination of such positivity is significant and expect that Theorem 1 will play a role in discrete spline interpolation, discrete minimization and other related topics.

REFERENCES

1. W. BOEHM, Inserting new knots into B-spline curves, *Computer-Aided Design* **12** (1980), 199–201.
2. C. DE BOOR, Splines as linear combinations of B-splines. A survey, *in* "Approximation Theory II" (G. G. Lorentz *et al.*, Eds.), pp. 1–47, Academic Press, New York, 1976.
3. C. DE BOOR, The total positivity of the spline collocation matrix, *Indiana Univ. Math. J.* **25** (1976), 541–552.
4. C. DE BOOR, "A Practical Guide to Splines," Springer-Verlag, New York, 1978.
5. E. COHEN, T. LYCHE, AND R. RIESENFELD, Discrete B-splines and subdivision techniques in computer-aided geometric design and computer graphics, *Computer Graphics and Image Processing* **14** (1980), 87–111.
6. S. KARLIN, "Total Positivity, I," Stanford Univ. Press, Stanford, Calif., 1968.
7. M. LANE AND R. RIESENFELD, A geometric proof for the variation diminishing property of B-spline approximation, *J. Approx. Theory* **37** (1983), 1–4.
8. T. LYCHE, "Discrete Polynomial Spline Approximation Methods," Ph.D. thesis, University of Texas, Austin, 1974.
9. O. L. MANGASARIAN AND L. L. SCHUMAKER, Discrete splines via mathematical programming, *SIAM J. Control Optim.* **9** (1971), 174–183.
10. I. J. SCHOENBERG AND A. WHITNEY, On Pólya frequency functions, III, *Trans. Amer. Math. Soc.* **74** (1953), 246–259.
11. L. L. SCHUMAKER, Constructive aspects of discrete polynomial spline functions, *in* "Approximation Theory" (G. G. Lorentz, Ed.), pp. 469–476, Academic Press, New York, 1973.